INCOMPLETE BLOCK DESIGNS

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1. Introduction

These designs were introduced by Yates in order to eliminate heterogeneity to a greater extent than is possible with randomized blocks and Latin squares when the number of treatments is large. The precision of the estimate of a treatment effect depends on the number of replications of the treatment - if larger is the number of replications, the more is the precision. Similar is the case for the precision of estimate of the difference between two treatment effects. If a pair of treatment occurs together more number of times in the design, the difference between these two treatment effects can be estimated with more precision. To ensure equal or nearly equal precision of comparisons of different pairs of treatment effects, the treatments are so allocated to the experimental units in different blocks of equal sizes such that each treatment occurs at most once in a block and it has an equal number of replications and each pair of treatments has the same or nearly the same number of replications. When the number of replications of all pairs of treatments in a design is the same, then we have an important class of designs called **Balanced Incomplete Block** (BIB) designs and when there are unequal number of replications for different pairs of treatments, then the designs are called as **Partially Balanced Incomplete Block** (PBIB) designs.

2. Balanced Incomplete Block (BIB) Designs

A BIB design is an arrangement of v treatments in b blocks each of size k (<v) such that

(i) Each treatment occurs at most once in a block
(ii) Each treatment occurs in exactly r blocks
(iii) Each pair of treatments occurs together in exactly λ blocks.

Example 2.1: A BIB design for v = b = 5, r = k = 4 and λ = 3 in the following:

Blocks	**(I)**	1	2	3	4
	(II)	1	2	3	5
	(III)	1	2	4	5
	(IV)	1	3	4	5
	(V)	2	3	4	5

The symbols v, b, r, k, λ are called the parameters of the design. These parameters satisfy the relations

$$vr = bk \qquad \ldots(2.1)$$

and

$$\lambda(v-1) = r(k-1) \qquad \ldots(2.2)$$

Each side of equation (2.1) represents the total number of experimental units or plots in the design. Equation (2.2) can be established by noting that a given treatment occurs with (k-1) other treatments in each of r blocks and also occurs with each of the other (v-1) treatments in λ blocks.

A BIB design cannot exist unless (2.1) and (2.2) are satisfied. For instance, no design exists for v = b = 6 and r = k = 3 since, from (2.2) λ=6/5 is not an integer. However, these conditions are not sufficient for the existence of a BIB design. Even if both (2.1) and (2.2) are satisfied, it does not follow that such a design exists. For example, no BIB design exits for v = 15, b = 21, r = 7, k = 5, and λ = 2 even though both conditions are satisfied. In search of a criterion for the availability of a BIB design, Fisher proved that no design with b<v is possible.

2.1 Construction of BIB Designs

There is no single method of constructing all BIB designs. Solutions of many designs are still **unknown**. We describe below a few well known series of BIB designs.

2.1.1 Unreduced BIB Designs

These designs are obtained by taking all combinations of the v treatments k at a time. Therefore, the parameters of all unreduced BIB designs are:

$$v,\ k,\ b = {}^{v}C_{k},\ r = {}^{v-1}C_{k-1},\ \lambda = {}^{v-2}C_{k-2}$$

The BIB design for v = 5 treatments given in the previous section is an example of an unreduced design. These unreduced designs usually require a large number of blocks and replications so that the resulting designs will often be too large for practical purposes.

2.1.2 BIB Designs using MOLS

Before we describe the method, we explain the concept of mutually orthogonal latin squares (MOLS) which will be used in the construction of BIB designs.

A latin square of order s is an arrangement of s symbols in an sxs array such that each symbol occurs once in each row and once in each column of the array. For example, the following are 4x4 latin squares of order 4 in symbols A, B, C, and D:

A	B	C	D		A	B	C	D		A	B	C	D
B	A	D	C		C	D	A	B		D	C	B	A
C	D	A	B		D	C	B	A		B	A	D	C
D	C	B	A		B	A	D	C		C	D	A	B

Two latin squares are pairwise orthogonal if, when one square is superimposed on the other, each symbol of one latin square occurs once with each symbol of the other square. Three or more squares are mutually orthogonal if they are pair-wise orthogonal. The three 4x4 latin squares above are mutually orthogonal.

A complete set of s-1 mutually orthogonal latin squares is known to exist for any $s = p^n$, where p is a prime number. Tables can be found in Fisher and Yates (1963). Now we describe the methods of constructing BIB designs using MOLS.

Suppose $v = s^2$ treatments are set out in an s x s array. A group of s blocks each of size s is obtained by letting the rows of the array represent blocks. Another group of s blocks is given by taking the columns of the array as blocks. Now suppose one of the orthogonal latin squares is superimposed on to the array of treatments. A further group of s blocks is obtained if all treatments common to a particular symbol in the square are placed in a block. Each of the s-1 orthogonal squares produces a set of s blocks in this manner. The resulting design is a BIB design with parameters $v=s^2$, $b=s^2+s$, $k=s$, $r=s+1$, $\lambda=1$.

Example 2.2: For $v = 3^2 = 9$ treatments a 3 x 3 array and a complete set of mutually orthogonal latin squares of order 3x3 are :

1 2 3	A B C	A B C
4 5 6	C A B	B C A
7 8 9	B C A	C A B

Four groups of 3 blocks are obtained from the rows, columns and the symbols of the two squares, as follows:

	(1 2 3)		(1, 4, 7)
Rows	(4 5 6)	Columns	(2, 5, 8)
	(7 8 9)		(3, 6, 9)
	(1, 5, 9)		(1, 6, 8)
First square	(2, 6, 7)	Second square	(2, 4, 9)
	(3, 4, 8)		(3, 5, 7)

It can be checked that this is a BIB design with parameters v=9, b=12, r=4, k=3, and $\lambda=1$.

2.1.3 Complementary Design

The complement of the design in Example 2.2 obtained by replacing treatments in a block by those which do not occur in the block, is the following:

(2, 5, 6, 7, 8, 9)	(2, 3, 5, 6, 8, 9)
(1, 2, 3, 7, 8, 9)	(1, 3, 4, 6, 7, 9)
(1, 2, 3, 4, 5, 6)	(1, 2, 4, 5, 7, 8)
(2, 3, 4, 6, 7, 8)	(2, 3, 4, 5, 7, 9)
(1, 3, 4, 5, 8, 9)	(1, 3, 5, 6, 7, 8)
(1, 2, 5, 6, 7, 9)	(1, 2, 4, 6, 8, 9)

The complementary is also a BIB design with parameters v=9, b=12, r=8, k=6, $\lambda= 5$. In general if, we have a BIB design with parameters v, b, r, k, λ then its complementary design is a BIB design with parameters $v' = v$, $b'=b$, $r' = b-r$, $k' = v-k$, $\lambda' = b-2r+\lambda$.

The complementary design of the design with parameters $v=s^2$, $b=s^2+s$, $k=s$, $r=s+1$, $\lambda=1$ obtained above will be a BIB design with parameters

$$v = s^2,\ b = s(s+1),\ r = s^2-1,\ k = s(s-1),\ \lambda = s^2 - s - 1.$$

The $s(s+1)$ blocks of the design for $v=s^2$ treatments have been arranged in $s+1$ groups of s blocks each. Now suppose a new treatment is added to all the blocks in a particular group and that the treatment added is different for each group; also, that one further block is added which consists entirely of these $s+1$ new treatments. This method produces a second series of BIB design with parameters $v = b = s^2 + s + 1$, $r = k = s + 1$, $\lambda = 1$. Its complement is also a BIB design with parameters $v=b=s^2+s+1$, $r=k=s^2$, $\lambda = s(s-1)$.

2.1.4 Symmetric BIB Designs

A BIB design in which v=b or r=k is called a symmetric BIB design. In symmetric BIB designs any two blocks have λ treatments in common.

2.1.5 α-Resolvable and Affine α-Resolvable Designs

It has been seen in section 2.1.2 above that the blocks of the designs $v=s^2$, $b = s^2+ 1$, $r=s+1$, $k=s$, $\lambda=1$ can be divided into (s+1) groups, each consisting of s blocks such that in each group each of the treatments is replicated once. That is, each group is a complete replicate. Such designs are called resolvable designs or 1-resolvable designs.

In general a BIB design is called α-Resolvable if its blocks can be divided into t groups each consisting of m blocks such that in each group every treatment appears exactly α times.

In addition to this, if any two blocks of the design belonging to the same group have a constant number of treatments in common, say q_1, and any two blocks belonging to different groups have a constant number of treatments in common, say q_2, then the design is called affine α-resolvable BIB design.

2.1.6 Dual Design

The dual of a BIB design with parameters v, b, r, k, λ is obtained by interchanging the treatment and block symbols in the original design. The parameters of the dual design are $v' = b$, $b' = v$, $r' = k$, $k'=r$. The dual of a BIB design is not always a BIB design. If the original design is a symmetrical BIB design, then its dual is also a BIB design with the same parameters.

2.1.7 Residual Design

In a synmmetric BIB design with parameters v=b, r=k, λ delete one block and also those treatments which appear in this (deleted) block from the remaining (b-1) blocks, the design so obtained is known as the residual design. The residual design is also a BIB design with parameters $v^* = v - k$, $b^*= b-1$, $r^*=r$, $k^* = k-\lambda$, λ.

2.1.8 Derived Design

By deleting any block of a symmetric BIB design with parameters v=b, r=k, λ and retaining all the treatments in b-1 blocks that appear in the deleted block, we obtain a BIB design which is called the derived design. The parameters of the derived design are $v'' = k$, $b'' = b-1$, r= r-1, k= λ, λ = λ-1.

2.2 Randomization Procedure

(i) Allot the treatment symbols (1,2,...,v) to the v treatments at random.
(ii) Allot the groups of k treatments to the b blocks at random.
(iii) Randomize the positions of the treatment numbers within each block.

2.3 Statistical Analysis

Consider the following model:

Observation = General mean + treatment effect + block effect + random error.

Random errors are assumed to be independently and identically distributed normally with mean zero and constant variance σ^2. On minimising the error sum of squares with respect to the parameters, we get a set of normal equations which can be solved to get the estimates of different contrasts of various treatment and block effects.

Now we compute

G = Grand total of observations

$\bar{y}$ = grand mean = G/n, where n= vr = bk = total number of observations

T_i = Sum of obervations for treatment i, (i=1,2,..., v)

B_j = Sum of observations in block j, (j=1,2,..., b)

$CF = G^2/n$,

Q_i = adjusted i^{th} treatment total

= T_i - (Sum of block totals in which treatment i occurs) / Block size (k)

A solution for the i^{th} treatment effect is,

$\hat{\tau} = (k\,Q_i) / (\lambda\, v)$ (i = 1,2, ..., v)

Adjusted treatment mean for treatment i= i^{th} treatment effects ($\hat{\tau}_i$) + grand mean ($\bar{y}$).

Various sums of squares can be obtained as follows:

(i) Total Sum of Squares (TSS) = Σ (observations)^2 - CF
(ii) Treatment Sum of Squares unadjusted (SST_u) = $[\Sigma\, T_i^2]/r - CF$
(iii) Block Sum of Squares unadjusted (SSB_U) = $[\Sigma\, B_j^2]\, /\, k - CF$
(iv) Treatments Sum of Squares adjusted (SST_A) = $\Sigma\, \hat{\tau}_i\, Q_i$
(v) Error SS (SSE) = TSS - SSB_U - SST_A
(vi) Blocks sum of squares adjusted (SSB_A) = $SST_A + SSB_U - SST_U$

The analysis of variance for a BIB design is given below:

Table 2.1: ANOVA for a BIB (v, b, r, k, λ) Design

Source	D.F.	SS	MS	F
Treatment (unadj.)	v-1	SST_u		
Blocks (unadjusted)	b-1	SSB_u		
Treatments (adjusted)	v-1	SST_A	MST	MST/MSE
Blocks adjusted	b-1	SSB_A	MSB	MSB/MSE
Error	n-b-v+1	SSE	MSE	
Total	n-1	TSS		

Note: $MST = SST_A / (v-1)$, $MSB = SSB_A / (b-1)$ and $MSE = SSE / (n - b - v + 1)$

Coefficient of Variation = $(\sqrt{MSE} / \bar{y}) \times 100$

Standard error of difference between two adjusted treatment means = $\left[2k\, MSE / (\lambda v)\right]^{1/2}$.

C.D. = $t_{0.05} \times \left[2k\, MSE / (\lambda v)\right]^{1/2}$

Exercise 2.1: The following data relate to an experiment conducted using a BIB design with parameters v=4, b=4, r=3, k=3, λ =2. The layout plan and yield figures (in coded units) are tabulated below:

Block Number	Treatments and yield figures		
1	(1) 77	(2) 85	(3) 60
2	(1) 70	(2) 67	(4) 54
3	(1) 69	(3) 62	(4) 40
4	(2) 72	(3) 63	(4) 55

Carry out the analysis.

Solution: Grand Total = G = 77 + 85+60 + ... + 55 = 774,

No. of observations = n = 12, Grand Mean = $\bar{y}$ = G/n = 64.5,

No. of Replications = r = 3, Block size = k = 3, C.F.= $G^2/n = \frac{599076}{12} = 49923$

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
Treat. /Block No.	(T_i)	(B_j)	Blocks No's in which Treat. i occurs	$\sum_{j(i)} B_j$	$\sum_{j(i)} B_j / k$	Q_i	$\hat{\tau}_i = kQ_i/\lambda v$ i^{th} treat. effect	Adj. treat. mean
1	216	222	1,2,3	584	194.67	21.33	8.0	72.50
2	224	191	1,2,4	603	201	23.00	8.63	73.13
3	185	171	1,3,4	583	194.33	- 9.33	-3.50	61.00
4	149	190	2,3,4	552	184	-35.00	-13.13	51.37

Treat. Total T_1 = 77+70+69 = 216, etc., Block Total B_1 = 77 +85+60 = 222, etc.
Total of blocks in which treat. i occurs $\sum_{j(i)} B_j$ = 222+191+171 = 584, etc.

Adj. treat. total (Q_i) = Ti - $\sum_{j(i)} B_j / k$ = T_i - 584/3= 21.33, etc.

Total S.S.(TSS) = $\Sigma(\text{observation})^2$ - CF=$77^2 + 85^2 + ... + 55^2$ - CF = 51442 - 49923 = 1519

Treatment S.S. unadj. (SST_u) = $(\Sigma T_i^2) / r$ - CF = 153258 - 49923 = 1163

Block S.S. unadj. (SSB_u) = $(\Sigma B_j^2) / k$ – CF = $\frac{151106}{3}$ - 49923 = 445.67

Treatment S.S. adj.(SST_A) = $\Sigma\, \hat{\tau}_i\, Q_i$ = 861.34

Error S.S. (SSE) = TSS-SSB_u - SST_A = 1519 - 445.67-861.34 = 211.99
Block S.S. adj.(SSB_A) = SST_A + SSB_u - SST_u = 861.34 + 445.67 – 1163 = 144.01

ANOVA

Source	**d.f.**	**S.S.**	**M.S.**	**F**
Blocks (unadj.)	3	445.67		
Treatments (adj.)	3	861.34	287.11	6.75
Blocks (adj.)	3	144.01	48.00	1.13
Treatments (unadj.)	3	1163.00		
Error	5	211.99	42.40	
Total	11	1519		

Table Value F (3,5) = 9.0135 (at 5% level of significance)
Treatment effects are not significant.

$$CD = t_{0.05} \times \left[2k\ MSE / (\lambda v)\right]^{1/2} = 2.57 \sqrt{\frac{2x3}{2x4} \times 42.40} = 14.4926$$

	$\overline{T_1}$	$\overline{T_2}$	$\overline{T_3}$	$\overline{T_4}$
Adjusted Treatment Means:	72.50	73.13	61.00	51.37

$\overline{T_1} - \overline{T_2}$ = -0.63, $\overline{T_1} - \overline{T_3}$ = 11.5
$\overline{T_1} - \overline{T_4}$ = 21.13 T_1 is significantly different from T_4
$\overline{T_2} - \overline{T_3}$ = 12.13
$\overline{T_2} - \overline{T_4}$ = 21.76 T_2 is significantly different from T_4
$\overline{T_3} - \overline{T_4}$ = 9.63,

3. Partially Balanced Incomplete Block (PBIB) Designs

BIB designs may not fit well to many experimental situations as these designs require a large number of replications. Moreover, these designs are not available for all numbers of treatments and block sizes. To overcome these difficulties PBIB designs were introduced. In these designs the variance of every estimated elementary contrast among treatment effects is not the same. The definition of PBIB designs is based on the association scheme.

Association Scheme

Given v treatment symbols 1,2,..,v, a relation satisfying the following conditions is called an m-class association scheme ($m \geq 2$).

(i) Any two symbols are either 1st, 2nd,..., or m^{th} associates; the relation of association being symmetric, i.e., if the symbol α is the i^{th} associate of β, then β is the i^{th} associate of α.

(ii) Each symbol α has n_i i^{th} associates, the number n_i being independent of α,

(iii) If any two symbols α and β are i^{th} associates, then the number of symbols that are j^{th} associates of α and k^{th} associate of β is p^i_{jk} and is independent of the pair of i^{th} associates α and β.

The numbers v, n_i and p^i_{jk} (i,j,k=1,2,...,m) are called the parameters of the association scheme and satisfy the following relations:

$$\sum_{i=1}^{m} n_i = v - 1$$

$$\sum_{k=1}^{m} p^i_{jk} = n_j - 1, \quad \text{if } i = j$$

$$= n_j, \quad \text{if } i \neq j$$

$$n_i p^i_{jk} = n_j p^j_{ik}$$

Example 3.1: Consider v=12 treatments denoted by numbers 1 to 12. Form 3 groups of 4 symbols each as follows: (1,2,3,4), (5,6,7,8), (9,10,11,12). We now define any two treatments as first associates if they belong to the same group, and second associates if they belong to the different groups. Here, $n_1 = 3$, $n_2 = 8$.

Definition: Given an association scheme with m classes ($m \geq 2$) we have a PBIB design with m associate classes based on the association scheme, if the v treatment symbols can be arranged into b blocks, such that

(i) Every symbol occurs at most once in a block.

(ii) Every symbol occurs in exactly r blocks.

(iii) If two symbols are i^{th} associates, then they occur together in λ_i blocks, the number λ_i being independent of the particular pair of i^{th} associates α and β.

The numbers v, b, r, k, λ_i (i=1,2,...,m) are called the parameters of the design. It can be easily seen that

$$vr = bk \text{ and } \sum_{i=1}^{m} n_i \lambda_i = r(k-1).$$

It may be mentioned that as in the case of BIB designs, the complementary design of a PBIB with parameters v,b,r,k,λ_i is also a PBIB design having the same association scheme with the parameters $v^*=v$, $b^*=b$, $r^*=b-r$, $k^*=v-k$, $\lambda_i^*=b-2r+\lambda_i$.

Two-class association schemes and the two-associate PBIB designs have been extensively studied in the literature and are simple to use. As an illustration, we describe Group Divisible (GD) association scheme and the designs based on it.

3.1 GD Association Scheme

Let v = mn symbols be arranged into m groups of n symbols each. A pair of symbols belonging to the same group is first associates [n_1 = n-1] and a pair of symbols belonging to different groups is second associates [n_2 = n(m-1)]. A PBIB (2) design based on a GD scheme is called a GD design.

3.2 Some GD Designs

Method 1: Let D be a BIB design with parameters v = m, b, r, k, λ. Obtain a design D^* from D by replacing the i^{th} treatment (i=1,2,...,v) in D by n new treatment symbols $i_1, i_2, ..., i_n$. D^* is a group divisible design with the following parameters v^*= mn, b^*= b, r^*= r, k^*= nk, m, n, λ_1 =r , λ_2=λ.

Example 3.1: Consider the following BIB design with parameters (4, 4, 3, 3, 2):

(1 2 3)
(1 2 4)
(1 3 4)
(2 3 4)

Replacing 1 by a, b; 2 by c, d; 3 by e, f and 4 by g, h, the following GD design with parameters v = 8, b = 4, r = 3, k = 6, λ_1 =3 , λ_2 = 2. is obtained:

(a b c d e f)
(a b c d g h)
(a b e f g h)
(c d e f g h)

Method 2: By omitting the blocks in which a particular treatment, say θ, occurs from a BIB design with the parameters v, b, r, k, λ =1, we obtain a GD design consisting of the remaining blocks with the parameters v^* =v-1, b^*=b-r, r^* = r-1, k^*=k, m=r, n=k-1, λ_1=0, λ_2= 1.

Exercise 3.1: A varietal trial on wheat crop was conducted using a two-associate class PBIB design. The parameters of the design are v = b = 9, r = k = 3, λ_1 = 1, λ_2 = 0, n_1 = 6, n_2 = 2. The data along with the block contents are given below: Analyse the data.

Blocks

I	59 (3)	56 (8)	53 (4)
II	35 (2)	33 (7)	40 (4)
III	48 (1)	42 (7)	42 (5)
IV	46 (7)	56 (8)	51 (9)
V	61 (4)	61 (5)	55 (6)
VI	52 (3)	53 (9)	48 (5)
VII	54 (1)	58 (8)	62 (6)
VIII	45 (2)	46 (9)	47 (6)
IX	31 (1)	27 (2)	35 (3)

Figures within brackets are treatment numbers.

4. Lattice Designs

Lattice designs form an important class of useful incomplete block designs. These designs were originally introduced by Yates (1936). Here, we shall deal with square lattices, rectangular lattices and cubic lattice designs.

4.1 Square Lattice Designs

The characteristic feature of these designs are that the number of treatments is a perfect square and the block size is the square root of this number. Moreover, incomplete blocks are combined in groups to form separate replications. The numbers of replications of the treatments are flexible in these designs and are useful for situations in which a large number of treatments are to be tested. If the design has two replications of the treatments, it is called a simple lattice; if it has 3 replications it is called a triple-lattice and so on. In general, if the number of replications is m, it is called an m-ple lattice. Square lattice designs can be constructed as follows:

Method of Construction

Let there be $v = s^2$ treatments, numbered as $1,2,\ldots, s^2$. Arrange these treatment numbers in the form of a s x s square which we call here as *standard array*. The contents of each of the s rows of this array are taken to form a block giving thereby s blocks each of size s. Further the contents of columns of this array are taken to form blocks giving another set of s blocks forming another complete replication. Next a s x s latin square is taken and is superimposed on the above standard array of treatment numbers. The treatment numbers that fall on a particular symbol of the latin square are taken to form a block. Thus we get s blocks corresponding to the s symbols of the latin square. Again, another latin square orthogonal to the previous one is taken and from this square also, another set of s blocks is obtained in the same manner. The process is repeated to get further replications. The process is continued till $m - 2$ ($\le s - 1$, if s is a prime or power of a prime) mutually orthogonal latin squares are utilized. When the complete set of (s-1) mutually orthogonal latin squares(if such a set exists) is utilized, the designs becomes a balanced (s + 1) - lattice. A balanced lattice is a BIB design.

Example 4.1: Let $v = 3^2 = 9$ and $k = 3$.

Standard Array

1	2	3
4	5	6
7	8	9

MOLS

A	B	C	A	B	C
B	C	A	C	A	B
C	A	B	B	C	A

Blocks

	1	(1,2,3)		7	(1,6,8)
Rep.I	2	(4,5,6)	**Rep.III**	8	(2,4,9)
	3	(7,8,9)		9	(3,5,7)
	4	(1,4,7)		10	(1,5,9)
Rep.II	5	(2,5,8)	**Rep.IV**	11	(2,6,7)
	6	(3,6,9)		12	(3,4,8)

All the four replications, in the above arrangement form a balanced lattice which is a BIB design with v = 9 , b = 12, r = 4, k = 3, λ = 1.

Randomisation
The randomisation consists of the following steps:
1. Allot the treatments to the treatment numbers at random.
2. Randomise the replications.
3. Randomise the blocks separately and independently within replication.
4. Randomise the treatments separately and independently within each block.

Steps 2 to 3 give each treatment and equal chance of being allotted to any experimental unit. These steps correspond to the allotment of treatments to units at random in an ordinary random choice of treatments which form the blocks of the design. If differences among blocks are large, the error variance per plot for the mean of a group of treatments that lie in the same block may be considerably higher than the average error variance. This additional randomisation ensures that the average error variance may be used, in nearly all cases, for comparisons among groups of treatments.

When a plan is repeated to obtain extra replications, a separate randomisation must be carried out for each additional replicate.

Exercise 4.1: The following table gives the synthetic yields per plot of an experiment conducted with 3^2=9 treatments using a simple lattice design.

Rep I				**Rep II**			
Blocks ↓	Treatments (yield per plot)			Blocks↓	Treatments (yield per plot)		
1	1 (8)	7 (5)	4 (3)	4	8 (2)	7 (2)	9 (7)
2	3 (3)	6 (2)	9 (6)	5	4 (3)	5 (3)	6 (3)
3	8 (3)	5 (7)	2 (3)	6	2 (2)	3 (4)	1 (6)

Analyse the data.

4.2 Rectangular Lattice Designs

Harshbarger (1947, 1949) developed rectangular lattices for s (s + 1) treatments in blocks of size s units. These designs form a useful addition to the square lattices. The statistical analysis of these designs is quite similar to that for simple and triple latices, though it takes more time because the block adjustments are not so simple as with square lattices. The new designs are less symmetrical than the square lattices, in the sense that there is a greater variation in the accuracy with which two treatment means are compared.

Method of Construction
There are several methods of constructing rectangular lattice designs we describe below one suggesting by G.S.Watson which uses a latin square with (s+1) rows and columns, in which every letter in the leading diagonal is different. When writing down the square we attach a number to all letters expect those in the leading diagonal, as illustrated below for a 4 x 4 square.

A	B1	C2	D3
D4	C	B5	A6
B7	A8	D	C9
C10	D11	A12	B

In the first replication we place in a block all numbers that lie in the same row of the latin square, in the second replication all numbers that lie in the same column, and in the third all numbers that have the same latin letter.

	Blocks	
	1	(1,2,3)
Rep.I	2	(4,5,6)
	3	(7,8,9)
	4	(1,4,7)
	5	(4,7,10)
Rep.II	6	(1,8,11)
	7	(2,5,12)
	8	(3,6,9)
	9	(6,8,12)
Rep.III	10	(1,5,7)
	11	(2,9,10)
	12	(3,4,11)

By using the first 2 replications from any plan we obtain a rectangular lattice in 2 replications, which by means of repetitions can be used for an experiment in 4,6,8, etc., replications. By using all 3 replications of the plan we have designs for 3,6,9, etc., replications.

4.3 Cubic Lattice Designs

Cubic lattice designs were introduced by Yates for plant breeding experiments in which selection are to be made from an unusually large number of varieties. The number of treatments must be an exact cube. The most useful range comprises 27, 64, 125, 216, 343, 512, 729, and 1000 treatments. The size of the block is the cube root of the number of treatments, i.e., 3,4,5,,6,7,8,9, and 10, respectively. Thus cubic lattices can accommodate a large number of treatments in a small size of incomplete block. The designs have been used, for example, in an experiment with 729 strains of ponderosa pine seedlings and an experiment with 729 soybean varieties. The number of replicates must be 3 or some multiple of 3.

To obtain a plan, the s^3 treatments are numbered by means of a three-digit code in which each digit takes all values from 1 to p. For 27 treatments, the codes are as given below:

Treatment No.	**Code**	**Treatment No.**	**Code**	**Treatment No.**	**Code**

1	111	4	121	7	131
2	211	5	221	8	231
3	311	6	321	9	331
10	112	13	122	16	132
11	212	14	222	17	232
12	312	15	322	18	332
19	113	22	123	25	133
20	213	23	223	26	233
21	313	24	323	27	333

The same principle applies with a large number of treatments. For the first s treatments, the last two digits are fixed at (11) while the first digit runs from 1 to s. The next s treatments are coded by fixing the last two digits at (21) while the first digit again runs from 1 to s, and so on in a systematic manner until the final s treatments are reached, for which the last two digits have the fixed values (ss). Within each of the 3 replications, the s^3 treatments are grouped into s^2 blocks, each of size s. In the first replication, the rule for this grouping is to keep the last two digits constant within a block, allowing the first digit to take values from 1 to s. Thus, in the example below, the 9 group of treatments constitute the 9 blocks. Block 1 containing the treatments (111), (211), and (311).

To form the blocks in the second replication, we keep the first and the last digits fixed within any block and given the second digit all values from 1 to s. With 27 treatments, the first block therefore contains (111), (121), and (131), the second block (211), (221), (231), and the last block (313), (323), and (333). In the third replication, the first and the second digits are constant within each block. The composition of the blocks in the second and third replication is shown below:

Replication-II

Treatment No.	Code	Treatment No.	Code	Treatment No.	Code
Block 1		***Block 2***		***Block 3***	
1	111	2	211	3	311
4	121	5	221	6	321
7	131	8	231	9	331
Block 4		***Block 5***		***Block 6***	
10	112	11	212	12	312
13	122	14	222	15	322
16	132	17	232	18	332
Block 7		***Block 8***		***Block 9***	
19	113	20	213	21	313
22	123	23	223	24	323
25	133	26	233	27	333

Replication-III

Treatment No.	Code	Treatment No.	Code	Treatment No.	Code

Block 1		**Block 2**		**Block 3**	
1	111	2	211	3	311
10	112	11	212	12	312
19	113	20	213	21	313
Block 4		**Block 5**		**Block 6**	
4	121	5	221	6	321
13	122	14	222	15	322
22	123	23	223	24	323
Block 7		**Block 8**		**Block 9**	
7	131	8	231	9	331
16	132	17	232	18	332
25	133	26	233	27	333

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