BLOCK DESIGNS WITH NESTED ROWS AND COLUMNS

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1. Introduction

For experimental situations where there are two cross-classified factors causing heterogeneity in the experimental material and are nested within the blocking factor, the block designs with nested rows and columns have been developed. Before giving definition and characterization of properties of these designs, let us have a look at the examples of some such experimental situations.

Experimental Situation 1: Consider the case of Agroforestry experiments involving evaluation of crop species/varieties for their comparative performance when grown along with a given tree species. The tree species would be adding to the heterogeneity in the growing conditions for the crop species / varieties being evaluated since (i) their stand (both number and vigour) in different plots (blocks) will be subjected to high variation and (ii) the tree row will have column (directional) effects and row (tree or tree row segment) effects nested within it. These row and column effects within a block are crossed between themselves.

Experimental Situation 2: Consider the case of animal nutrition experiments where number of lactations has been taken as blocking factor. However, age and stage of lactation within animals of same number of lactations may contribute significantly to error variance and these two factors are crossed with each other. Therefore, for such situations, within blocking factor, two cross-classified factors, age (rows) and stage of lactation (columns) are nested.

Experimental Situations 3: Consider an experiment, which was conducted using a block design. The experiment was harvested blockwise. To meet the objectives of the experiment, the harvested samples are to be analysed for their contents in the laboratory by different technicians over different time periods. The factors, technicians and time periods are cross-classified with each other. Therefore, to control the variations due to technicians and time periods, this has to be taken as a situation of block designs with nested rows (technicians) and columns (time periods).

Therefore, in this kind of situation the experimental units are broadly classified into ‘*b*’ blocks such that within each block the experimental units can be arranged in ‘*p*’ rows and ‘*q*’ columns. Therefore, we have a total of *bpq* experimental units.

To cope with the above type of situations, repeated lattice - square designs, (Cochran and Cox, 1957, pp. 483-497,Yates1940) were introduced, where each square can be considered a block (complete) within which are nested two other factors, denoted by ‘rows’ and ‘columns’, so that one can eliminate two sources of variation within each block. However, the repeated lattice squares have some limitations viz.

1. Available only for $v = s^2$ treatments when s is a prime or power ;
2. Complete Block is required.

Srivastava (1978) generalised this idea by giving a variance balanced design with $v = 5$ in (2×2) squares - a design when $v \neq s^2$.

$v = 5, b = 5, r = 2, p = 2, q = 2, n = 20$

1 2	2 3	3 4	4 5	5 1
3 4	4 5	5 1	1 2	2 3

The information matrix for treatment comparisons for this design is

$$\mathbf{C} = (5/4)\,(\mathbf{I} - (1/5)\,\mathbf{11}').$$

Singh and Dey (1979) generalized the idea of repeated lattice - square designs to a class of designs, called Block Designs with Nested Rows and Columns. They termed these designs as Balanced Incomplete Block Designs with Nested Rows and Columns (BIB-RC Design). They gave definition, properties and a method of analysis of these designs. Several methods of construction of these designs were also given.

2. BIB-RC Designs

Definition 1: A block design with nested rows and columns with v treatments and b sets (blocks), each block containing p rows and q columns ($pq<v$) is said to be a BIB-RC design if the following conditions are satisfied:

(i) every treatment occurs at most once in a block;

(ii) given a pair of treatments(i,j)

$$p\lambda_{r(i,j)} + q\lambda_{c(i,j)} - \lambda_{b(i,j)} = \lambda \text{ (constant)}$$

where $\lambda_{r(i,j)}$, $\lambda_{c(i,j)}$ and $\lambda_{b(i,j)}$ denote the number of blocks in which treatment i and j occur together in the same row, same column and elsewhere respectively and λ is a constant independent of i and j.

It is easy to see that in a BIB-RC design, every treatment occurs in exactly r blocks, where

$$r = \lambda\,(v-1)/\{(p-1)(q-1)\}.$$

Two methods of construction of BIB-RC designs have been described by Singh and Dey (1979). These are described below.

Construction of BIB-RC Designs

Method 2.1: If d is a prime or prime power, a BIB design with parameters $v = d^2$, $b = d(d+1)$, $r = d+1$, $k = d$, $\lambda = 1$ exists. This design can be split into ($d+1$) sets of d rows and d columns each such that the set of {rows} alone or {columns} alone form a BIB design.

Example 2.1: For d=3, a BIB design with parameters $v = 9$, $b = 12$, $r = 4$, $k = 3$, $\lambda = 1$ exists. Then using the above method we get a BIB-RC Design with parameters $v = 9$, $b = 4$, $p = 3$, $q = 3$, $r = 4$, $\lambda = 1$. The arrangement is as follows:

Block-1			Block-2			Block-3			Block-4		
1	2	3	1	4	7	1	6	8	1	9	5
4	5	6	2	5	8	9	2	4	6	2	7
7	8	9	3	6	9	5	7	3	8	4	3

Consider now another BIB design with parameters $v', b', r', k' = d^2, \lambda'$. Then, using the above arrangement of the BIBD $\{d^2, d\,(d+1, d,1)\}$, we get a BIB-RC design. This leads to the following result.

Theorem 2.1: The existence of a BIB design $\{v', b', r', k'= d^2, \lambda'\}$, d being a prime or prime power, implies the existence of a BIB-RC design with parameters

$$v = v', s = b'(d+1), r = r'(d+1), p = q = d, \lambda = (d-1)\ \lambda'.$$

To illustrate the above result, consider the following example.

Example 2.2: Let $d = 2$. The BIB design ($v = 4, b = 6, r = 3, k = 3, \lambda = 1$) can be resolved into three sets such that the {rows} and {columns}, treated as blocks form a BIB design each. The arrangement is as shown below.

1 2	1 3	1 4
3 4	4 2	2 3

Example 2.3: Consider the BIB design ($v' = 7 = b', r' = 4 = k', \lambda' = 2$), the blocks for this design being obtained from the initial block (1, 4, 6, 7) mod 7. Writing the arrangement above with the block contents of each of the blocks of the second BIB design, we get the following BIB-RC design.

1 4	1 6	1 7	2 5	2 7	2 1	3 6	3 1	3 2
6 7	7 4	4 6	7 1	1 5	5 7	1 2	2 6	6 1
4 7	4 2	4 3	5 1	5 3	5 4	6 2	6 4	6 5
2 3	3 7	7 2	3 4	4 1	1 3	4 5	5 2	2 4
7 3	7 5	7 6						
5 6	6 3	3 5						

The above design has parameters $v = 7, b = 21, r = 12, p = q = 2, \lambda = 2.$

Method 2.2: Suppose there exists a BIB design, for which a solution based on initial blocks is available. Suppose further that these initial blocks can be arranged in rows and columns in such a manner that the differences arising from rows and columns from all the initial blocks are symmetrically repeated. Then, by developing these initial sets of rows and columns, we can get a BIB-RC design. The following example illustrates this construction method.

Example 2.4: Consider a BIB design with parameters $v = 13, b = 26, r = 12, k = 6, \lambda = 5$, a solution of which can be obtained by developing the initial blocks (1,3,9,4,10,12) and (2,5,6,7,8,11) mod 13. Now, arrange blocks into 2 rows and 3 columns as under:

1	3	9		7	8	11
12	10	4		6	5	2

Developing these two initial row-column sets, we obtain a solution for the BIB-RC design with parameters

$$v = 13, b = 26, r = 12, p = 2, q = 3, \lambda = 2.$$

Several other construction methods of BIB-RC designs have been proposed by several authors in literature. For an excellent review on these methods of construction, one way refer to Sreenath (1998).

The case of the complete block designs, with $pq = v$, was considered by Cheng (1986) and were called as Balanced Complete Block Designs with nested Rows and Columns with same parameters and denoted as BCBRCD (v, b, r, p, q, λ). Concept of BIB-RC designs was also extended to partially balanced incomplete block designs with nested rows and columns as well, repeated lattice square designs fall in this category. Some methods of construction of BIB-RC designs have also been given where rows (alone) or columns (alone) does not form a BIB design. The example given by Srivastava (1978) belongs to this case.

3. Analysis of Block Designs with nested rows and columns

In its most general form, suppose v treatments are to be compared via n experimental units arranged in b blocks such that j^{th} block is of size $k_j = p_j q_j$; $\forall\ j=1(1)b$, and there are p_j rows and q_j columns in the j^{th} block. Let n_{ijlm} denote the number of times i^{th} treatment is applied in the m^{th} column and l^{th} row of j^{th} block; $\forall\ m=1(1)q_{dj};\ l=1(1)p_{dj}$; j = (1)b; $i=1(1)v$.

Let us consider the following matrices for obtaining reduced normal equations.

$\mathbf{N} = ((n_{ij..}))$: $v \times b$ incidence matrix of treatments Vs blocks;

$\mathbf{N_1} = ((n_{ijl.}))$: $v \times \sum_{j=1}^{b} p_{dj}$ incidence matrix of treatments Vs rows;

$\mathbf{N_2}=((n_{ij.m}))$:~v× $\sum_{j=1}^{b} q_{dj}$ incidence~matrix~of~treatments~Vs~columns.

$\mathbf{Q} = \sum_{j=1}^{b}{}^{+} q_{dj}\mathbf{I}_{p_{dj}}, \mathbf{P} = \sum_{j=1}^{b}{}^{+} p_{dj}\mathbf{I}_{q_{dj}}$ denote respectively the $\sum_{j=1}^{b} p_{dj} \times \sum_{j=1}^{b} p_{dj}$ and $\sum_{j=1}^{b} q_{dj} \times \sum_{j=1}^{b} q_{dj}$ diagonal matrices of row sizes and column sizes.

$\mathbf{K} =$ Diag($p_{d1}q_{d1}, \ldots, p_{db}q_{db}$), is the $b \times b$ diagonal matrix of block sizes.
$\mathbf{R}$ = Diag($r_{d1},\ldots,r_{dv}$), where r_{di}, $i=1(1)v$ is the replication number of i^{th} treatment.

The reduced normal equations are $\mathbf{C}\tau = \mathbf{Q}$, where τ is the $v \times 1$ vector of treatment effects. The coefficient matrix $\mathbf{C}$ of reduced normal equations for estimating linear functions of treatment effects using a block design with nested rows and columns is

$$\mathbf{C}_d = \mathbf{R}_d - \mathbf{N}_{d1}\mathbf{Q}_d^{-1}\mathbf{N}'_{d1} - \mathbf{N}_{d2}\mathbf{P}_d^{-1}\mathbf{N}'_{d2} + \mathbf{N}_d\mathbf{K}_d^{-1}\mathbf{N}'_d . \tag{3.1}$$

$$\mathbf{C}_d = \mathbf{R}_d - \mathbf{N}_{d2}\mathbf{P}_d^{-1}\mathbf{N}'_{d2} - \mathbf{L}^*(= \mathbf{N}_{d1}\mathbf{Q}_d^{-1}\mathbf{N}'_{d1} - \mathbf{N}_d\mathbf{K}_d^{-1}\mathbf{N}'_d).$$

$\mathbf{C}_d$ is symmetric non-negative definite matrix with rows and columns sums zero and for a connected design Rank ($\mathbf{C}_d$) = v-1. The $v \times 1$ vector of adjusted totals is

$\mathbf{Q} = \mathbf{T} - \mathbf{N}_{d_1}\mathbf{Q}_d^{-1}\mathbf{M} - \mathbf{N}_{d_2}\mathbf{P}_d^{-1}\mathbf{L} + \mathbf{N}_d\mathbf{K}_d^{-1}\mathbf{B}$, where $\mathbf{T}_{(v\times 1)}$, $\mathbf{M}_{\left(\sum p_{dj}\times 1\right)}$, $\mathbf{L}_{\left(\sum q_{dj}\times 1\right)}$, $\mathbf{B}_{(b\times 1)}$ note respectively the vectors of totals of treatments, rows, columns and blocks. The adjusted treatment sum of squares is given by $\mathbf{Q}'\mathbf{C}^{-}\mathbf{Q}$. For proper block design set up i.e., when $k = pq$

$$\mathbf{C}_d = \mathbf{R}_d - (1/q)\mathbf{N}_{d1}\mathbf{N}'_{d1} - (1/p)\mathbf{N}_{d2}\mathbf{N}'_{d2} + (1/pq)\mathbf{N}_d\mathbf{N}'_d. \quad (3.2)$$

$$\mathbf{C}_d = \mathbf{R}_d - (1/p)\mathbf{N}_{d2}\mathbf{N}'_{d2} - \mathbf{L}(=(1/q)\mathbf{N}_{d2}\mathbf{N}'_{d2} - (1/pq)\mathbf{N}_d\mathbf{N}'_d. \quad (3.3)$$

For a BIB-RC design $\mathbf{C}_d = \frac{\lambda v}{pq}\left(\mathbf{I} - \frac{1}{v}\mathbf{1}\mathbf{1}'\right)$. Therefore for a BIB-RC design the adjusted treatment sum of squares is $\frac{pq}{\lambda v}\sum_{i=1}^{v} Q_i^2$.

4. Variance-Balanced Designs

Definition 4.1: A block design with nested rows and columns is said to be variance balanced if the design estimates each elementary contrast of treatment effects with the same variance.

Theorem 4.1: A necessary and sufficient condition for block design with nested rows and columns to be variance balanced is that its **C** matrix is of the form

$$\mathbf{C} = \theta(\mathbf{I}_v - v^{-1}\mathbf{1}_v\mathbf{1}'_v) \quad (4.1)$$

where θ is the unique non-zero eigenvalue of **C** with multiplicity (v-1).

For a BIB-RC design $\theta = (\lambda v)/(pq) = b(p-1)(q-1)/(v-1)$. In binary and proper setting $c_{ii} = r_i(p-1)(q-1)/pq$ and the condition **C1** = **0** and constancy of off diagonal elements for a BIB-RC design ensures that all r_i's are equal. Therefore in the binary and proper setting $c_{ii} = r(p-1)(q-1)/pq$ and hence trace is fixed for a BIB-RC design. Therefore in the class of binary block designs with nested rows and columns, a BIB-RC design or a BCB-RC design, whenever existent is universally optimal.

However, in a general class of block designs with nested rows and columns D (v, b, p, q), BIB-RC design may not be optimal. To be more clear, consider an experimental situation where 4 treatments are to be compared via 24 experimental units arranged in 6 blocks such that there are 2 rows and 2 columns in each of the blocks. From a number of designs belonging to this class of designs let us choose the following two designs.

Design 1: BIB-RC Design ($v = 4, b = 6, r = 6, p = 2, q = 2, \lambda = 2$)

1 2	1 3	1 4	1 2	1 3	1 4
3 4	4 2	2 3	3 4	4 2	2 3

$\mathbf{C}$ = (8/4) ($\mathbf{I}$ - (1/4)$\mathbf{1}\mathbf{1}'$). Var($\hat{\tau}_i - \hat{\tau}_j$) = σ^2 $\forall\, i \neq j = 1,2,3,4$

Design 2: Variance Balanced Designs with nested rows and columns Design ($v = 4, b = 6, r = 6, p = 2, q = 2, \lambda = 2$)

1 2	1 3	1 4	2 3	2 4	3 4
2 1	3 1	4 1	3 2	4 2	4 3

$\mathbf{C} = (4)\,(\mathbf{I} - (1/4)\ \mathbf{11}')$. $\text{Var}(\hat{\tau}_i - \hat{\tau}_j) = \sigma^2/2 \ \ \forall\ i \neq j = 1,2,3,4$

Now we see that in over class of connected proper block designs with nested rows and columns BIB-RC designs are not optimal.

Various optimality aspects of block designs with nested rows and columns were studied by Bagchi, Mukhopadhyay and Sinha (1990), Chang and Notz (1988), Chang and Notz (1989), Chang and Notz (1990) and Morgan and Udin (1993) over D (v, b, p, q). They termed variance-balanced designs with nested rows and columns as Balanced Nested Row-Column Designs (BN-RC designs).

5. Balanced Nested Row -Column design

Definition 5.1: A block design in 'v' treatments arranged in 'b' blocks such that each block contains'p' rows and 'q' columns is said to be a BN-RC design if following (i) and (ii) conditions of theorem are satisfied *i.e.* if

(i) $\mathbf{L} = \mathbf{N}_1\mathbf{N}_1' - \frac{1}{p}\,\mathbf{NN}' = \mathbf{O}$

(ii) $\mathbf{N}_{d_2}$ is the incidence matrix of a Balanced Block Design.

Whenever a BN-RC design d $\in$ D (v, b, p, q) exists, it is universally optimal.

If $pq \leq v$ *i.e.* BIB-RC exists, then $p \leq v$ and hence, for a BN-RC design with $pq \leq v$,

$$\mathbf{C}_1 = \frac{\lambda v}{p}\left(\mathbf{I}_v - \frac{1}{v}\mathbf{1}_v\mathbf{1}_v'\right) = \frac{bq(p-1)}{v-1}\left(\mathbf{I}_v - \frac{1}{v}\mathbf{1}_v\mathbf{1}_v'\right)$$

If BIB-RC design exists

$$\mathbf{C}_2 = \frac{\lambda v}{pq}\left(\mathbf{I} - \frac{1}{v}\mathbf{11}'\right) = \frac{b(p-1)(q-1)}{v-1}\left(\mathbf{I} - \frac{1}{v}\mathbf{11}'\right)$$

$$\mathbf{C}_1 = \frac{q}{q-1}\mathbf{C}_2$$

Relative efficiency of BIB-RC design as compared to an BN-RC design is = $(q-1)/q$

Theorem 5.2: A block design with nested rows and columns for 'v' treatments, 'b' blocks of p-rows and q-columns is universally optimal if

1. the columns form a variance balanced (Balanced block) design.
2. a treatment appears in a row of a block it appears equally frequently.

In such a setting design is a block design, with columns as blocks and all the results available for block design setting are applicable to block designs with nested rows and columns provided if a treatment appears in a row of a block, it appears equally frequently in all rows of that block.

Constructions of BN-RC Design

Method 5.3: Arrange the contents of each of the blocks of a BIB design v, b, r, k, λ, in the form of a latin square. We get a BN - RC design which is universally optimal over D ($v, b, p = k, q = k$).

Example 5.1: Consider a BIB design D (4,6,3,2,1) with block contents as

(1,2), (1,3), (1,4), (2,3), (2,4), (3,4).

Following the procedure above we get a BN - RC design with *parameters* $v = 4, b = 6, p = 2, q = 2$. The layout of the design is as given below:

1 2	1 3	1 4	2 3	2 4	3 4
2 1	3 1	4 1	3 2	4 2	4 3

Method 5.2: Existence of a BIB design with parameters v, b, r, k, λ and a Youden Square design YSD (k, p). Then on arranging the contents of each of the blocks of BIBD in the form of a Youden Square, we get a BN - RC design ($v, b, p, q = k$).

Example 5.3: Let a BIB design D (4, 4, 3, 3, 2) with block contents as
(1, 2, 3); (1, 2, 4); (1, 3, 4); (2, 3, 4).

Let a Youden Square design (3, 2) exists.
A B C
B C A.

Following the procedure of method 3.2, we get a BN - RC design with parameters $v = 4, b = 4, p = 2, q = 3$.

1 2 3	1 2 4	1 3 4	2 3 4
2 3 1	2 4 1	3 4 1	3 4 2

Remark: A BN - RC design with parameters $v, b = v(v-1)/2, \quad p = 2 = q$ always exists. The BN-RC design with parameters $v = 5, b = 10, r = 8, p = 2, q = 2$ is

1 2	1 3	1 4	1 5	2 3	2 4	2 5	3 4	3 5	4 5
2 1	3 1	4 1	5 1	3 2	4 2	5 2	4 3	5 3	5 4

The optimality aspects of these designs under non-proper block design setup have been studied by Parsad, Gupta and Voss (2001). Several methods of construction of optimal non-proper block designs with nested rows and columns have been given by Chakraborty (1996) and Parsad, Gupta and Voss (2001).

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