FRACTIONAL FACTORIAL PLANS

Aloke Dey
I.S.I., 7, SJS Sansanwal Marg, New Delhi – 110 016
adey@isid.ac.in

1. Introduction

Factorial experiments are widely used in agriculture and several other branches of science. Typically, in such an experiment there is an output variable, which is dependent on several controllable or input variables. These input variables are called *factors*. For each of the factors, there are two or more possible settings known as *levels*. Any combination of the levels of all the factors under consideration is called a *treatment combination*. For example, consider an experiment where n factors are at levels $s_1, s_2, \cdots, s_n, s_i \geq 2 \; \forall \; i = 1,2,\cdots,n$, respectively. All the $s_1 x s_2 x \cdots x s_n$ possible combinations of the levels of the n factors form the treatment combinations. Factorial experiments aim at exploring the effects of the individual factors and perhaps their inter-relationship as well.

A factorial experiment, where each treatment combination is applied to at least one experimental unit, is called a *complete* factorial. However, quite often in practice, the total number of treatment combinations is too large to allow the use of a complete factorial. Since each factor involves at least two levels, this can happen even with only a moderate number of factors. For example, consider an agronomic experiment where the objective is to study the effects of nutrients (factors) on the yield of paddy. Even if only three major nutrients, namely, nitrogen, phosphorus and potash and seven micronutrients, namely, boron, copper, iron, magnesium, manganese, molybdenum and zinc, are included in the experiment, and each of these appears at only two levels, namely presence or absence, there will be as many as $2^{10} = 1024$ treatment combinations. The use of a complete factorial will then involve at least *1024* experimental units. The break up of the degrees of freedom for the data generated from a single replication, assuming that only interactions up to *3* factors are important, would be,

Source	**DF**
Main effects	10
Two-factor interactions	45
Three-factor interactions	120
Error	848
Total	1023

In this experiment it is not possible to get the estimate of the error. Since the higher order interactions are often not important and are generally difficult to interpret, these can be used for estimating the error. The error degrees of freedom, however is unduly large (*848).* Such a large experiment, apart from being infeasible from considerations of cost and time, may not at all be necessary if the interest is in estimating the lower order effects, under the assumption of absence of higher order effects. Moreover, in planning such big experiments non-experimental types of error may also creep in. These errors may arise because of mishandling of the big experiment in the sense that the treatment labelling may be changed or plot numbers may be wrongly noted.

Thus, unless the number of factors is too small, quite commonly in practice complete factorials are not affordable and one has no other alternative but to include only some but not all treatment combinations in a factorial experiment. Obviously, the experiment then includes only a fraction of the totality of all possible treatment combinations and therefore, the underlying experimental strategy is called a *fractional factorial plan* or *fractional replication.* Therefore, t**he technique of recovering useful information with a reasonable degree of precision by observing only a part of the complete**

factorial is known as fractional replication, a concept introduced by Finney[1] (1945). Such a plan aims at drawing, under appropriate assumptions, valid statistical inferences about the relevant factorial effects through an optimal utilization of the available resources. In the present write-up, we discuss some preliminary aspects of fractional factorial designs.

2. Some Preliminaries

In this section, we explain some preliminaries related to a fractional factorial plan and, to that end, we think it appropriate to introduce the ideas by means of a simple example. Suppose we have three factors, say *A*, *B*, *C*, each at two levels. Clearly there are *8* treatment combinations *viz. 000, 001, 010, 100, 011, 101, 110, 111* in this experiment. Suppose the experimenter has resources to conduct an experiment with only *4* of the possible *8* treatment combinations. This calls for consideration of a $\frac{1}{2}$ fraction of a 2^3 experiment that can be obtained by confounding the interaction *ABC* and taking one of the two blocks. Let the following *4* treatment combinations be chosen for the experiment:

$$100,\ \ 010,\ \ 001,\ \ 111.$$

Suppose y_1, y_2, y_3 and y_4 are the observations corresponding to these four treatment combinations. Then, under a standard model, the expected values of these four observations are

$$IE(y_1) = \mu + A - B - C - AB - AC + BC + ABC$$
$$IE(y_2) = \mu - A + B - C - AB + AC - BC + ABC$$
$$IE(y_3) = \mu - A - B + C + AB - AC - BC + ABC$$
$$IE(y_4) = \mu + A + B + C + AB + AC + BC + ABC$$

where *IE* stands for the expectation, μ is the general mean and $A, AB,...$ denote the main effects and interactions. Clearly, one cannot estimate all the eight parameters, *viz*, $\mu, A, B, C, AB, AC, BC, ABC$ from only four observations. In fact, it can be seen that

$$IE\left\{\frac{1}{4}(y_4 + y_1 - y_2 - y_3)\right\} = A + BC\ ;$$
$$IE\left\{\frac{1}{4}(y_4 - y_1 + y_2 - y_3)\right\} = B + AC\ ;$$
$$IE\left\{\frac{1}{4}(y_4 - y_1 - y_2 + y_3)\right\} = C + AB\ ;$$
$$IE\left\{\frac{1}{4}(y_4 + y_1 + y_2 + y_3)\right\} = \mu + ABC\ .$$

Thus, neither the general mean, nor any factorial effect can be unbiasedly estimated, *unless* some *a priori* assumptions are made in the form of absence of certain factorial effects. For example, if there is a reason to believe (either from past experience or from the underlying physical process) that the three-factor interaction *ABC* has negligible magnitude, then one can get an unbiased estimator of μ. Similarly, all main effects are estimable through the $\frac{1}{2}$ fraction considered here if all the two factor interactions are assumed to be absent.

The kind of mixing up of different factorial effects demonstrated in the above example is a common feature of all fractional factorial designs. As a matter of fact, in any fractional factorial designs, linear

[1]Finney, D.J. (1945). The fractional replication of the factorial arrangements. *Ann. Eugen.* **12,** 291-301.

combinations of two or more factorial effects are estimated by an observational contrast. All factorial effects that appear in the expectation of an observational contrast are said to be *aliased* with each other. For instance, in the above example, A is aliased with BC, B with AC *etc.* The general mean is aliased with the three-factor interaction ABC and we express this in the form of an algebraic identity

$$I \equiv ABC\,.$$

This relation is known as the *defining contrast* or the *identity relationship* and determines the type of fraction one is choosing. Given the identity relationship, one can get the alias structure of each of the factorial effects, in the case of fractions of 2^n factorials, by

(i) interpreting I as unity and,

(ii) formally multiplying each effect on both sides of the identity relationship and deleting any letter whose power is two.

Thus, continuing with the example described above, we have

$$A = A(ABC) = A^2BC = BC$$
$$B = B(ABC) = AB^2C = AC$$
$$C = C(ABC) = ABC^2 = AB$$

where the equality sign between the two factorial effects is to be interpreted as "*aliased with*". One may argue here that the experiment could have been conducted with the treatment combination *000, 011, 101, 110*. In that case, following above, it can easily be seen that

$$IE(y_1) = \mu - A - B - C + AB + AC + BC - ABC$$
$$IE(y_2) = \mu - A + B + C - AB - AC + BC - ABC$$
$$IE(y_3) = \mu + A - B + C - AB + AC - BC - ABC$$
$$IE(y_4) = \mu + A + B - C + AB - AC - BC - ABC$$

Clearly, one cannot estimate all the eight parameters, *viz,* $\mu, A, B, C, AB, AC, BC, ABC$ from only four observations. In fact, it can be seen that

$$IE\left\{\frac{1}{4}(y_4 + y_3 - y_2 - y_1)\right\} = A - BC\,;$$
$$IE\left\{\frac{1}{4}(y_4 - y_3 + y_2 - y_1)\right\} = B - AC\,;$$
$$IE\left\{\frac{1}{4}(-y_4 + y_3 + y_2 - y_1)\right\} = C - AB\,;$$
$$IE\left\{\frac{1}{4}(y_4 + y_1 + y_2 + y_3)\right\} = \mu + ABC\,.$$

Therefore, we can see that this half replicate also reduces to same state of affairs and does not make much difference practically, which of the two half replicates is considered.

If instead of the chosen fraction with identity relation $I \equiv ABC$, one had considered a fraction with identity relation $I \equiv AB$, then the fraction would consist of the following treatment combinations

$$100,\quad 010,\quad 011,\quad 101\,.$$

The alias structure for this fraction would be

$$A = B$$
$$C = ABC$$
$$AC = BC\,.$$

However, this is not an appropriate fraction as, in this case two main effects appear as aliases of each other and hence, even under the assumption of absence of all interactions, this fraction is not capable of providing unbiased estimators of all main effects, which, are invariably the effects of interest.

The above discussion has brought out several basic issues related to a fractional factorial design:

(i) a fractional factorial design results in loss of information that is contained in a complete factorial;

(ii) the fraction has to be chosen carefully to minimize the loss;

(iii) assumptions about the absence of certain factorial effects (generally the higher order factorial effects) have to be made in order to get unbiased estimates of lower order effects.

3. More Complex Fractions

We now discuss fractions, which are more "*complex*" than the one considered in Section 2. In general let us consider the problem of constructing a $1/2^k$ fraction of a 2^n factorial experiment. Such an experiment will be denoted by 2^{n-k}. Of the total of (2^n - 1) effects and interactions in the full factorial experiment, 2^k-1 will be inseparable from the mean, and the remaining 2^n - 2^k will be mutually inseparable in sets of 2^k, there being (2^{n-k} - 1) such sets. The treatment combinations will be selected to be of the same sign as the control in the interactions X_1, X_2, ..., X_k will also be inseparable from the mean, and the identity relationship will become

$$I = X_1 = X_2 = X_1X_2 = X_3 = X_1X_3 = X_2X_3 = X_1X_2X_3 = \text{etc.} \tag{3}$$

The alias sets of an effect or interaction Y are the generalized interactions of Y with X_1, X_2, X_1X_2, etc.

Example 3: Consider a 2^8 factorial and let the factors be denoted by $A, B, ..., H$. The complete 2^8 factorial involves 256 treatment combinations. Suppose it is desired to have a $\frac{1}{4}\left(=\frac{1}{2^2}\right)$ fraction of a 2^8 factorial. We need to have three factorial effects in the defining contrast; out of which two are '*independent*' and the third is the generalized interaction of these two. One may choose the identity relation as

$$I = ABCDE = ABFGH = CDEFGH .$$

If one now determines the alias structures of all effects, it can be seen that no main effect is aliased with a factorial effect with less than four factors; similarly, no two-factor interaction is aliased with another two-factor interaction. For example,

$$A \;\; = BCDE = BFGH \;\; = ACDEFGH$$
$$AC = BDE \;\; = BCFGH = ADEFGH .$$

The actual construction of the fraction (*i.e.*, identification of the treatment combinations in the fraction) can be achieved as follows: Let $x_1, x_2, ..., x_8$ be variables representing the levels of the factors $A, B, ..., H$ respectively, i.e., each x_i takes either the value 0 or 1. The treatment combinations for the fraction with identity relation $I = ABCDE = ABFGH = CDEFGH$ can be obtained by solving the equations

$$x_1 + x_2 + x_3 + x_4 + x_5 = 0$$
$$x_1 + x_2 + x_6 + x_7 + x_8 = 0$$

simultaneously; the arithmetic to be followed for solving the equations is that every even number is to be replaced by 0 and every odd number by 1. For example, let us take

$$x_1 = 1 = x_2 = x_3 = x_4 = x_6 = x_7 , \quad x_5 = x_8 = 0 .$$

This choice of x_i's does satisfy the two equations above and is thus a solution of these equations. Hence the treatment combination 11110110 is a member of the fraction.

Example 4: As a further illustration, consider a fraction of the factorial experiment (2^6, 2^4) obtained by confounding *ABCD, ABEF, and CDEF*. The 16 treatment combinations in the fraction are

000000,110000, 001100, 111100, 0000011, 110011, 001111, 111111, 011010, 101010, 010110, 100110, 011001, 101001, 010101, 100101

The defining contrast is

$$I \equiv ABCD \equiv ABEF \equiv CDEF, \tag{4}$$

and the alias sets in this case are

$A \equiv BCD \equiv BEF \equiv ACDEF$,	$B \equiv ACD \equiv AEF \equiv BCDEF$,
$C \equiv ABD \equiv ABCEF \equiv DEF$,	$D \equiv ABC \equiv ABDEF \equiv CEF$,
$E \equiv ABCDE \equiv ABF \equiv CDF$,	$F \equiv ABCDF \equiv ABE \equiv CDE$,
$AB \equiv CD \equiv EF \equiv ABCDEF$,	$AC \equiv BD \equiv BCEF \equiv ADEF$,
$AD \equiv BC \equiv BDEF \equiv ACEF$,	$AE \equiv BCDE \equiv BF \equiv ACDF$,
$AF \equiv BCDF \equiv BE \equiv ACDE$,	$CE \equiv ABDE \equiv ABCF \equiv DF$,
$CF \equiv ABDF \equiv ABCE \equiv DE$,	$ACE \equiv BDE \equiv BCF \equiv ADF$,
$ACF \equiv BDF \equiv BCE \equiv ADE$.	

3.1. Design Resolution

A fractional factorial design is of resolution *R* if no *p*-factor effect is aliased with another effect containing less than *R - p* factors. This definition is due to Box and Hunter[2] (1961).

We usually employ a Roman numeral subscript to denote design resolution; thus, the one-half fraction of the 2^3 design with the defining relation $I = ABC$ (or $I = -ABC$) is a 2^{3-1}_{III} design.

Designs of resolution *III*, *IV* and *V* are particularly important. The definition of these designs and an example of each follows:

1. Resolution *III* designs. These are designs, in which no main effects are aliased with any other main effect, but main effects are aliased with two-factor interactions and two-factor interactions may be aliased with each other. The 2^{3-1} design considered earlier with treatment combinations *100, 010, 011, 111*of resolution *III* (2^{3-1}_{III}).
2. Resolution *IV* designs. These are designs in which no main effect is aliased with any other main effect or with any two-factor interactions, but two-factor interactions are aliased with other two factor interactions. A 2^{4-1} design with $I = ABCD$ is of resolution *IV* (2^{4-1}_{IV}).
3. Resolution *V* designs. These are designs in which no main effect or two-factor interaction is aliased with any other main effect or two-factor interaction, but two-factor interactions are aliased with three-factor interactions. A 2^{5-1} design with $I = ABCDE$ is of resolution *V* (2^{5-1}_{V}). The example of quarter replicate of 2^8 factorial is a resolution *V* plan.

In general, the resolution of a two-level fractional factorial design is equal to the smallest number of letters in any word in the defining relation. Consequently, we could call the preceding design types

[2] Box, G.E.P. and Hunter, J.S. (1961). The 2^{k-p} fractional factorial plans. **Technometrics**, **3**, ***311-352***.

three-letter, four-letter, and five-letter designs, respectively. We usually like to employ fractional designs that have the highest possible resolution consistent with the degree of fractionation required. The higher the resolution, the less restrictive the assumptions that are required regarding which interactions are negligible in order to obtain a unique interpretation of the data.

This fraction is called a Resolution-*V* fraction, since the lowest order factorial effect in the identity relation is a five-factor interaction. (In contrast to this, the fraction considered in Section 2 with identity relation $I \equiv ABC$ is a Resolution-III fraction, since a three- factor interaction appears in the identity relation).

The above definition is applicable only to the regular fractional factorial plans of symmetrical factorials (by a regular fraction we mean a fraction whose treatment combinations form a sub-group). Therefore, Webb[3] (1968) has given a generalized definition of resolution to cover all types of fractional replications. A fractional factorial plan is of resolution $2R$ if it permits the estimation of all effects upto $R - 1$ factor interactions, when all effects involving $R + 1$ factors and more are assumed as negligible and it is said to be of resolution $2R + 1$ if it permits the estimation of all effects upto R-factor interactions, when all effects involving $R + 1$ factors are assumed to be zero.

All regular fractional factorial plans for symmetrical factorials provide estimates of the relevant effects without correlation. A fractional factorial plan that permits the estimation of all relevant effects with zero correlation is called an orthogonal plan. A class of fractional factorial plans that is very popular with the experimenters and is useful for exploratory experiments is the class of *main effects* plan. These designs are capable of ensuring the estimability of the mean and all the main effects under the assumption that all interactions are absent. These designs are often highly fractionated and some times require just the minimal number of runs (equal to the number of parameters to be estimated). Such plans are known as saturated plans. If the main effect plans are orthogonal, these are known as orthogonal main effect plans. We give below three examples of such main effect plans.

Example 3: For a 2^7 factorial, the following *8* runs constitute an orthogonal main effect plan:
1111111, 0101010, 0011001, 1110000, 0100101, 1000011, 0010110,1001100

This fraction permits the estimability of the mean and all the main effects; furthermore, the least squares estimates of these parameters are mutually uncorrelated.

Example 4: The following is an orthogonal main effect plan for a 3^4 experiment in *9* treatment combinations:
0000, 1101, 2202, 1210, 2011, 0112, 2120, 0221, 1022.

Example 5: Factorial experiments where not all factors appear at the same number of levels are called asymmetric factorials. Suppose we have one factor at four levels and four factors at two levels each. Such an experiment is denoted by 4×2^4 factorial experiment. One can verify that the following *8* treatment combinations constitute an orthogonal main effect plan for a 4×2^4 experiment:
00000,01111,10011,11100,20101,21010,30110,31001.

Here the first factor has four levels, coded as *0, 1, 2, 3* while the remaining factors are at two levels each, coded as *0, 1*.

[3] Webb, S.R. (1968). Non-orthogonal designs for even resolution. **Technometrics**, **10**, ***535-550***.

3.2 Fractional Replication in Time (Sequential Programme of Experiments)

Fractional factorial plans may result into saving of efforts in the lines of work where an experiment can be conducted one part at a time. Suppose that a complete replication of a 2^4 experiment is planned and *8* treatment combinations can be processed at one time. Divide the complete replicate into two half replicates of *8* treatment combinations each. In the fractional factorial approach, we stop when the first half replicate has been completed and decide from the results whether to continue with the second block or to change the direction of the experimentation. If *A* alone show a definite response, we may decide that the further experimentation with these four factors is unjustified. On the other hand, if *A*, *B*, and the alias pair $AD = BC$ all give marked effects, we complete the replicate in order to obtain independent estimates of *AD* and *BC* so as to infer upon the nature of these interactions.

Sometimes the blocking is required within the fractional replication.

What we have described above is merely an introduction to the theory of fractional factorial designs. We have mainly restricted attention to symmetrical factorials, *i.e.*, those factorials in which each factor appears at the same number of levels. The problems concerning asymmetric fractional factorials are generally more complex in nature. There are many interesting and challenging problems that emerge in the study of fractional designs. The literature on fractional factorial designs is voluminous and continuous to grow. In fact, this is currently one of the most active areas of research. For elementary ideas about fractional factorials, the reader is refereed to Hinkelmann and Kempthorne[4] (1994). More advanced description relating to orthogonal fractions (that is, those which permit uncorrelated estimates of relevant parameters) is available in Dey[5] (1985) and for a mathematical treatment of fractional factorial designs, including optimality aspects and recent developments, the reader may refer to Dey and Mukerjee[6] (1998). To summarize the fractional factorials can be useful in the following real life situations

i) Situations where some of the higher order interactions can be assumed to be zero.
ii) In screening from a large set of factors.
iii) In a sequential programme of experiments.
iv) In variance estimation in complex surveys.

4. Analysis of Fractional Factorials

The analysis of fractional factorials is similar to the analysis of the full factorials. The treatment groups for each main effect or interaction are found by solving appropriate sets of equations and then the sum of squares are obtained from the observation totals of these treatment groups by the usual method.

For 2^n factorials, the fractionally replicated designs can also be analysed by applying Yate's algorithm. The only difference is that while writing the treatments, levels of *k* factors have to be ignored in the case of $1/2^k$ fraction. These *k* factors are so chosen that as a result of their suppression no treatment combination of the remaining $n - k$ factors should have zeros only, or repeat. The other $n - k$ factors are introduced one by one while writing the treatments, as in full replication. Here, $n - k$ columns will be generated by following the same method as described in full factorial. An interaction corresponding to a contrast is also found similarly by considering only the $(n - k)$ factors. The rest of the interactions that will contain the ignored factors also will form aliases of the above interactions.

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[4] Hinkelmann, K. and O. Kempthorne (1994). ***Designs and Analysis of Experiments Vol. I.*** Wiley: New York.

[5] Dey, A. (1985). ***Orthogonal Fractional Factorial Designs***. Halsted Press: New York.

[6] Dey, A. and R. Mukerjee (1998). ***Fractional Factorial Plans***. Wiley: New York.

The fractions of 3^n factorials can also be analysed on the same analogy. Here, also, while writing the treatments, factors are suppressed first and then they are written by introducing the factors one by one as described in full factorial. The operations and the correspondence of contrasts and interactions are also similar when the non-suppressed factors alone are considered. It is, however, not possible to write the aliases of such interaction components. But this does not create any serious problem. The linear and quadratic contrasts for a suppressed factor, *L*, come from contrasts involving those with which *L* is in alias.

Illustration: An exploratory trial on Cardamom was conducted in Madras state with seven factors each at two levels in one quarter of a replicate. The design was a confounded one with 16 plots per block. The treatments were all combinations of presence and absence of Zinc, Copper, Boron, Iron, Manganese, Magnesium and Molybdenum denoted by *A, B, C, D, E, F* and *G* respectively. The plot size was six plants in a row. The doses in *lb per acre* were *20, 20, 80, 224, 20, 10* and *100* for Zinc, Copper, Manganese, Magnesium, Molybdenum, Boron and Iron, respectively. The layout plan and the yield of green pods in *100 gr. per plot* are given below.

Table 1

Block 1		**Block 2**	
ab	15	1	3
ae	4	*be*	11
cd	7	*abcd*	6
bcde	12	*acde*	4
adf	13	*bdf*	10
abdef	15	*def*	4
bcf	17	*acf*	9
cef	5	*abcef*	1
bdg	6	*adg*	8
fg	14	*abdeg*	1
befg	9	*bcg*	6
abcdfg	12	*ceg*	10
acdefg	11	*aefg*	5
deg	5	*cdfg*	2
acg	13	*bcdefg*	1
abceg	6	*abfg*	5

The identity group of interactions for the above $1/4(2^7)$ factorial is

$$I = ABCDE = CDFG = ABEFG.$$

The interactions confounded for the two blocks are

$$BCEF = ADF = BDEG = ACG.$$

The data have been analysed by using Yate's algorithm. For this purpose two factors have to be suppressed in the available treatment combinations. These two factors should be such that after suppression no treatment combination is repeated. In the present case *E* and *G* are two such factors. They have, thus, been suppressed. These have been shown in bracket while writing the treatment combinations in the table of analysis.

The sum of squares in the last column of Table 2 corresponds to the treatment combinations of the non-suppressed factors only as written in the first column. Those three or four factor interactions which have either a main effect or two factor interaction in its alias group are shown along with the main effect or interaction in the last column. There are thus 6 interactions none of which is in alias with a main effect or two factor interaction. One of them is confounded. The remaining five have been pooled together and used as main effect or two-factor interaction. One of them is confounded. The remaining five have been pooled together and used as error sum of squares. The error mean square has thus come out as *10.85*. When tested against this mean square only the interaction *BG* comes out to be significant. The average yields in the *BG* table are the following:

	b_0	b_1
g_0	*6.2*	*11.2*
g_1	*8.5*	*5.8*

The table indicates that application of Copper or Molybdenum alone proved better while their combined application depressed the yield.

Table 2

Treat. Comb.	**Obsn.**	**Columns**					**S.S.**	**Effects**
		1	**2**	**3**	**4**	**5**		
1	3	7	33	68	117	250		
a(e)	4	26	35	49	133	6	1.12	*A*
b(e)	11	23	20	65	-3	16	8.00	*B*
ab	15	12	29	68	9	-28	24.50	*AB*
c(eg)	10	13	33	8	9	-6	1.12	*C*
ac(g)	13	7	32	-11	7	-2	0.12	*AC*
bc(g)	6	11	42	-25	-11	-16	8.00	*BC*
abc(eg)	6	18	26	34	-17	-20	12.50	*ABC=DE*
d(eg)	5	19	5	8	11	-16	8.00	*D*
ad(g)	8	14	3	1	-17	46	50.00	*AD*
bd(g)	6	14	-2	-1	-9	2	0.12	*BD*
abd(g)	1	18	-9	8	7	2	0.12	*ABD=CE*
cd	7	17	-13	0	-17	-8	2.00	*CD*
acd(e)	4	25	-12	-11	1	0	0.00	*ACD=BE*
bcd(e)	12	13	14	-15	-1	26	21.12	*BCD=AE*
abcd	6	13	20	-2	-19	42	55.12	*ABCD=E*
f(g)	14	1	19	2	-19	16	8.00	*F*
af(eg)	5	4	-11	9	3	12	4.50	*AF*
bf(eg)	9	3	-6	-1	-19	-2	0.12	*BF*
abf(g)	5	0	7	-16	59	-6	1.12	*ABF=EG*
cf(e)	5	3	-5	-2	-7	-28	24.50	*CF*
acf	9	-5	4	-7	9	16	8.00	*ACF*
bcf	17	-3	8	1	-11	18	10.12	*BCF*
abcf(g)	1	-6	0	6	13	-18	10.12	*ABCF*
df(e)	4	-9	3	-30	7	22	15.12	*DF*
adf	13	-4	-3	13	-15	78	190.12	*ADF(Con)*
bdf	10	4	-8	9	-5	16	8.00	*BDF*
abdf(e)	15	-16	-3	-8	5	24	18.00	*ABDF*
cdf(g)	2	9	5	-6	43	-22	15.12	*CDF=G*
acdf(eg)	11	5	-20	5	-17	10	3.12	*ACDF=AG*
bcdf(eg)	1	9	-4	-25	11	-60	112.50	*BCDF=BG*
abcdf(g)	12	11	2	6	31	20	12.50	*ABCDF=EF*