1. Introduction

In Incomplete block designs, as their name implies, the block size is less than the number of treatments to be tested. These designs were introduced by Yates in order to eliminate heterogeneity to a greater extent than is possible with randomized blocks and Latin squares when the number of treatments is large. The precision of the estimate of a treatment effect depends on the number of replications of the treatment - if larger is the number of replications, the more is the precision. Similar is the case for the precision of estimate of the difference between two treatment effects. If a pair of treatment occurs together more number of times in the design, the difference between these two treatment effects can be estimated with more precision. To ensure equal or nearly equal precision of comparisons of different pairs of treatment effects, the treatments are so allocated to the experimental units in different blocks of equal sizes such that each treatment occurs at most once in a block and it has an equal number of replications and each pair of treatments has the same or nearly the same number of replications. When the number of replications of all pairs of treatments in a design is the same, then we have an important class of designs called Balanced Incomplete Block (BIB) designs. Thus a BIB design, an arrangement of \( v \) treatments in \( b \) blocks each of size \( k \) (\(<v\)) such that

- (i) Each treatment occurs at most once in a block
- (ii) Each treatment occurs in exactly \( r \) blocks
- (iii) Each pair of treatments occurs together in exactly \( \lambda \) blocks.

Example: A BIB design for \( v=b=5, r=k=4 \) and \( \lambda = 3 \) in the following:

<table>
<thead>
<tr>
<th>Blocks</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(I)</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>(II)</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>(III)</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>(IV)</td>
<td>1</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>(V)</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
</tbody>
</table>

The symbols \( v, b, r, k, \lambda \) are called the parameters of the design. These parameters satisfy the relations

\[
vr = bk \quad \text{(1.1)}
\]

\[
\lambda(v-1) = r(k-1) \quad \text{(1.2)}
\]

Each side of equation (1.1) represents the total number of experimental units or plots in the design. Equation (1.2) can be established by noting that a given treatment occurs with \( (k-1) \) other treatments in each of \( r \) blocks and also occurs with each of the other \( (v-1) \) treatments in \( \lambda \) blocks.
A balanced incomplete block design cannot exist unless (1.1) and (1.2) are satisfied. For instance, no design exists for \( v = b = 6 \) and \( r = k = 3 \) since, from (1.2) \( \lambda = 6/5 \) is not an integer. However, these conditions are not sufficient for the existence of a BIB design, even if both (1.2) and (1.3) are satisfied it does not follow that such a design exists. For example, no BIB design exits for \( v = 15, b = 21, r = 7, k = 5, \) and \( \lambda = 2 \) even though both conditions are satisfied. In search of a criterion for the availability of a BIB design Fisher proved that no design with \( b < v \) is possible.

2. Construction

There is no single method of constructing all BIB designs. Solutions of many designs are still unknown. We describe below a few well-known series of BIB designs.

2.1 Unreduced BIB designs

These designs are obtained by taking all combinations of the \( v \) treatments \( k \) at a time. Therefore, the parameters of all unreduced BIB designs are:

\[
v, k, b = vC_k, r = v^{-1}C_{k-1}, \lambda = v^{-2}C_{k-2}
\]

The BIB design for \( v = 5 \) treatments given in the previous section is an example of an unreduced design.

These unreduced designs usually require a large number of blocks and replications so that the resulting designs will often be too large for practical purposes.

2.2 BIB designs with parameters

\[
v = s^2, b = s^2 + s, k = s, r = s + 1, \lambda = 1
\]

Before we describe the method, we explain the concept of mutually orthogonal latin squares which will be used in the construction of BIB designs.

A latin square of order \( s \) is an arrangement of \( s \) symbols in an \( s \times s \) array such that each symbol occurs once in each row and once in each column of the array. For example, the following are 4x4 latin squares of order 4 in symbols A, B, C, and D:

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>A</td>
<td>D</td>
<td>C</td>
</tr>
<tr>
<td>C</td>
<td>D</td>
<td>A</td>
<td>B</td>
</tr>
<tr>
<td>D</td>
<td>C</td>
<td>B</td>
<td>A</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>D</td>
<td>C</td>
<td>B</td>
<td>A</td>
</tr>
<tr>
<td>C</td>
<td>D</td>
<td>A</td>
<td>B</td>
</tr>
<tr>
<td>B</td>
<td>A</td>
<td>D</td>
<td>C</td>
</tr>
</tbody>
</table>

Two latin squares are pairwise orthogonal if, when one square is superimposed on the other, each symbol of one latin square occurs once with each symbol of the other square. Three or more squares are mutually orthogonal if they are pair-wise orthogonal. The three 4x4 latin squares above are mutually orthogonal.

A complete set of \( s-1 \) mutually orthogonal latin squares is known to exist for any \( s = p^n \), where \( p \) is a prime number. Tables can be found in Fisher and Yates (1963). Now we describe the methods of constructing BIB designs with parameters given at (2.2)
Suppose \( v = s^2 \) treatments are set out in an \( s \times s \) array. A group of \( s \) blocks each of size \( s \) is obtained by letting the rows of the array represent blocks. Another group of \( s \) blocks is given by taking the columns of the array as blocks. Now suppose one of the orthogonal latin squares is superimposed on to the array of treatments. A further group of \( s \) blocks is obtained if all treatments common to a particular symbol in the square are placed in a block. Each of the \( s-1 \) orthogonal squares produces a set of \( s \) blocks in this manner. The resulting design is a BIB design with parameters \((2, 2)\).

Example 2. For \( v = 3^2 = 9 \) treatments a \( 3 \times 3 \) array and a complete set of mutually orthogonal latin squares of order \( 3 \times 3 \) are:

\[
\begin{array}{ccc|ccc|ccc}
1 & 2 & 3 & A & B & C & A & B & C \\
4 & 5 & 6 & C & A & B & B & C & A \\
7 & 8 & 9 & B & C & A & C & A & B \\
\end{array}
\]

Four groups of 3 blocks are obtained from the rows, columns and the symbols of the two squares, as follows:

- Rows: \((1, 2, 3), (4, 5, 6), (7, 8, 9)\)
- Columns: \((1, 4, 7), (2, 5, 8), (3, 6, 9)\)
- First square: \((1, 2, 3), (2, 6, 7), (3, 4, 8)\)
- Second square: \((1, 5, 9), (2, 4, 9), (3, 5, 7)\)

It can be checked that this is a BIB design with parameters \( v = 9, b = 12, r = 4, k = 3, \) and \( \lambda = 1. \)

**Complementary Design**

The complement of the design in Example 2 obtained by replacing treatments in a block by those which do not occur in the block, is the following:

\[
\begin{array}{cccccccc}
2 & 5 & 6 & 7 & 8 & 9 & 2 & 3 & 5 & 6 & 8 & 9 \\
1 & 2 & 3 & 7 & 8 & 9 & 1 & 3 & 4 & 6 & 7 & 9 \\
1 & 2 & 3 & 4 & 5 & 6 & 1 & 2 & 4 & 5 & 7 & 8 \\
2 & 3 & 4 & 6 & 7 & 8 & 2 & 3 & 4 & 5 & 7 & 9 \\
1 & 3 & 4 & 5 & 8 & 9 & 1 & 3 & 5 & 6 & 7 & 8 \\
1 & 2 & 5 & 6 & 7 & 9 & 1 & 2 & 4 & 6 & 8 & 9 \\
\end{array}
\]

The complementary design is also a BIB design with parameters \( v = 9, b = 12, r = 8, k = 6, \lambda = 5. \)

In general if, we have a BIB design with parameters \( v, b, r, k, \lambda \) then its complementary design is a BIB design with parameters \( v' = v, b' = b, r' = b - r, k' = v - k, \lambda' = b - 2r + \lambda. \)

The complementary design of the design with parameters \((2, 2)\) will be a balanced incomplete block design with parameters.
\( v = s^2, b = s(s+1), r = s^2-1, k = s(s-1), \lambda = s^2 - s -1. \)  \hspace{1cm} (2.3)

The \( s(s+1) \) blocks of the design for \( v=s^2 \) treatments have been arranged in \( s+1 \) groups of \( s \) blocks each. Now suppose a new treatment is added to all the blocks in a particular group and that the treatment added is different for each group; also, that one further block is added which consists entirely of these \( s+1 \) new treatments. This method produces a second series of BIB design with parameters

\( v = b = s^2 + s + 1, \quad r = k = s + 1, \quad \lambda = 1. \)  \hspace{1cm} (2.4)

Its complement is also a BIB design with parameters

\( v = b = s^2+s+1, \quad r = k = s^2, \quad \lambda = s(s-1) \) \hspace{1cm} (2.5)

**Symmetric BIB Designs**

A BIB design in which \( v = b \) or \( r = k \) is called a symmetric BIB design. The BIB design with parameters given at (2.4) and (2.5) are symmetric BIB designs. In symmetric BIB designs any two blocks have \( \lambda \) treatments in common.

**\( \alpha \)-Resolvable and Affine \( \alpha \)-Resolvable Designs**

It has been seen above that the blocks of the designs \( v = s^2, b = s^2 + 1, r = s+1, k = s, \lambda = 1 \) can be divided into \( (s+1) \) groups, each consisting of \( s \) blocks such that in each group each of the treatments is replicated once. That is, each group is a complete replicate. Such designs are called resolvable designs or 1-resolvable designs.

In general a BIB design is called \( \alpha \)-Resolvable if its blocks can be divided into \( t \) groups each consisting of \( m \) blocks such that in each group every treatment appears exactly \( \alpha \) times.

In addition to this, if any two blocks of the design belonging to the same group have a constant number of treatments in common, say \( q_1 \), and any two blocks belonging to different groups have a constant number of treatments in common, say \( q_2 \), then the design is called affine \( \alpha \)-resolvable BIB design.

**Dual Design**

The dual of a BIB design with parameters \( v, b, r, k, \lambda \) is obtained by interchanging the treatment and block symbols in the original design. The parameters of the dual design are \( v' = b, \quad b' = v, \quad r' = k, \quad k' = r. \) The dual of a BIB design is not always a BIB design. If the original design is a symmetrical BIB design, then its dual is also a BIB design with the same parameters.

**Residual Design**

In a symmetric BIB design with parameters \( v = b, \quad r = k, \quad \lambda \) delete one block and also those treatments which appear in this (deleted) block from the remaining \( (b-1) \) blocks, the
design so obtained is known as the residual design. The residual design is also a BIB design with parameters $v^* = v-k$, $b^* = b-1$, $r^* = r$, $k^* = k-\lambda$, $\lambda$.

**Derived Design**

By deleting any block of a symmetric BIB design with parameters $v = b$, $r = k$, $\lambda$ and retaining all the treatments in $b-1$ blocks that appear in the deleted block, we obtain a BIB design which is called the derived design. The parameters of the derived design are $v'' = k$, $b'' = b-1$, $r' = r-1$, $k = \lambda$, $\lambda = \lambda - 1$.

**Randomization Procedure**

(i) Allot the treatment symbols $(1, 2, ..., v)$ to the $v$ treatments at random.
(ii) Allot the groups of $k$ treatments to the $b$ blocks at random.
(iii) Randomize the positions of the treatment numbers within each block.

3. **Statistical Analysis**

Consider the model:

Observation = General mean + treatment effect + block effect + random error.

Random errors are assumed to be independently and identically distributed normally with mean zero and constant variance $\sigma^2$. On minimizing the error sum of squares we get a set of normal equations which can be solved to get the estimates of different contrasts of various estimated treatment and block effects.

Now we compute

\[
G = \text{Grand total of observations} \\
\bar{y} = \text{grand mean} = G/n, \text{ where } n = vr = bk = \text{total number of observations} \\
T_i = \text{Sum of observations for treatment } i, \ (i=1,2,...,v) \\
B_j = \text{Sum of observations in block } j, \ (j=1,2,...,b) \\
CF = G^2/n, \\
Q_i = \text{adjusted } i^{th} \text{ treatment total} \\
= T_i - (\text{Sum of block totals in which treatment } i \text{ occurs}) / \text{Block size (k)}
\]

The $i^{th}$ treatment effect,

\[
\hat{\tau}_i = (k Q_i) / (\lambda \ v) \\
i = 1,2, ..., v
\]

Adjusted treatment mean for treatment $i = i^{th}$ treatment effects ($\tau_i$) + grand mean ($\bar{y}$).

Various sums of squares can be obtained as follows:

(1) Total Sum of Squares (TSS) = $\Sigma$ (observations)$^2$ - CF
(2) Treatment Sum of Squares unadjusted (SSTu) = $[\Sigma T_i^2] / r$ - CF
(3) Block Sum of Squares unadjusted (SSBu) = $[\Sigma B_j^2] / k$ - CF
(4) Treatments Sum of Squares adjusted (SSTA) = $\Sigma \hat{\tau}_i Q_i$
The analysis of variance for a BIB design is given below:

### Table 1: ANOVA for a BIB Design

<table>
<thead>
<tr>
<th>Source</th>
<th>D.F.</th>
<th>SS</th>
<th>MS</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Treatment (unadj.)</td>
<td>v-1</td>
<td>SSTₚ</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Blocks (unadjusted)</td>
<td>b-1</td>
<td>SSBₚ</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Treatments (adjusted)</td>
<td>v-1</td>
<td>SSTₐ</td>
<td>MST</td>
<td>MST/MSE</td>
</tr>
<tr>
<td>Blocks adjusted</td>
<td>b-1</td>
<td>SSBₐ</td>
<td>MSB</td>
<td>MSB/MSE</td>
</tr>
<tr>
<td>Error</td>
<td>n-b-v+1</td>
<td>SSE</td>
<td>MSE</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>n-1</td>
<td>TSS</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Note:** MST = SSTₐ / (v-1), MSB = SSBₐ / (b-1)

and MSE = SSE / (n - b - v + 1)

Coefficient of variation = (\(\sqrt{\text{MSE}} / \bar{\text{y}}\)) x 100

Standard error of difference between two adjusted treatment means = \[2k \times \frac{\text{MSE}}{(\lambda v)^{1/2}}\].

C.D. = \(t_{0.05} \times [2k \times \frac{\text{MSE}}{(\lambda v)^{1/2}}]\)

**Example:** The following data relate to an experiment conducted using a BIB design with parameters \(v = 4\), \(b = 4\), \(r = 3\), \(k = 3\), \(\lambda = 2\). The layout plan and yield figures (in coded units) are tabulated below:

<table>
<thead>
<tr>
<th>Block Number</th>
<th>Treatments and yield figures</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(1) 77</td>
</tr>
<tr>
<td>2</td>
<td>(1) 70</td>
</tr>
<tr>
<td>3</td>
<td>(1) 69</td>
</tr>
<tr>
<td>4</td>
<td>(2) 72</td>
</tr>
</tbody>
</table>

| (2)        | (3) 85                      |
| (4)        | (3) 67                      |
| (4)        | (3) 62                      |
| (4)        | (4) 54                      |
| (4)        | (4) 50                      |

Carry out the analysis.

Grand Total = \(G = 77 + 85 + 60 + \ldots + 55 = 774\)

No. of observations = \(n = 12\)

Grand Mean = \(\bar{y} = G/n = 64.5\)

No. of Replications = \(r = 3\)

Block size = \(k = 3\)

C.F. = \(\frac{G^2}{n} = \frac{599076}{12} = 49923\)
B.C. Advanced Incomplete Block Designs

<table>
<thead>
<tr>
<th>Treat./Block No.</th>
<th>(T&lt;sub&gt;i&lt;/sub&gt;)</th>
<th>(B&lt;sub&gt;j&lt;/sub&gt;)</th>
<th>Blocks No’s in which Treat. i occurs</th>
<th>∑&lt;sub&gt;j(i) B&lt;sub&gt;j&lt;/sub&gt;&lt;/sup&gt;</th>
<th>∑&lt;sub&gt;j(i) B&lt;sub&gt;j&lt;/sub&gt;/k&lt;/sup&gt;</th>
<th>Q&lt;sub&gt;i&lt;/sub&gt;</th>
<th>i=1/kQ&lt;sub&gt;i&lt;/sub&gt;/λv i&lt;sup&gt;th&lt;/sup&gt; treat. effect</th>
<th>Adj. treat. mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>216</td>
<td>222</td>
<td>1,2,3</td>
<td>584</td>
<td>194.67</td>
<td>21.33</td>
<td>8.0</td>
<td>72.50</td>
</tr>
<tr>
<td>2</td>
<td>224</td>
<td>191</td>
<td>1,2,4</td>
<td>603</td>
<td>201</td>
<td>23.00</td>
<td>8.63</td>
<td>73.13</td>
</tr>
<tr>
<td>3</td>
<td>185</td>
<td>171</td>
<td>1,3,4</td>
<td>583</td>
<td>194.33</td>
<td>-9.33</td>
<td>-3.50</td>
<td>61.00</td>
</tr>
<tr>
<td>4</td>
<td>149</td>
<td>190</td>
<td>2,3,4</td>
<td>552</td>
<td>184</td>
<td>-35.00</td>
<td>-13.13</td>
<td>51.37</td>
</tr>
</tbody>
</table>

Treat. Total T<sub>1</sub> = 77 + 70 + 69 = 216, etc.
Block Total B<sub>1</sub> = 77 + 85 + 60 = 222, etc.
Total of blocks in which treat. i occurs ∑<sub>j(i) B<sub>j</sub></sup> = 222 + 191 + 171 = 584, etc.
Adj. treat. total (Q<sub>i</sub>) = T<sub>i</sub> - ∑<sub>j(i) B<sub>j</sub>/k</sup> = T<sub>i</sub> - 584/3 = 21.33, etc.

Total S.S.(TSS) = \( \sum (\text{observation})^2 - \text{CF} \)
= 77<sup>2</sup> + 85<sup>2</sup> + ... + 55<sup>2</sup> - CF
= 51442 - 49923 = 1519

Treatment S.S. unadj. (SST<sub>u</sub>) = (\( \sum T_i^2 \)) / r - CF
= 153258 - 49923 = 1163

Block S.S. unadj. (SSB<sub>u</sub>) = (\( \sum B_j^2 \)) / k - CF
= \( \frac{151106}{3} \) - 49923 = 445.67

Treatment S.S. adj.(SST<sub>A</sub>) = \( \sum \hat{\tau}_i Q_i \)
= 861.34

Error S.S. (SSE) = TSS-SSB<sub>u</sub> - SST<sub>A</sub>
= 1519 - 445.67 - 861.34 = 211.99

Block S.S. adj.(SSB<sub>A</sub>) = SST<sub>A</sub> + SSB<sub>u</sub> - SST<sub>u</sub>
= 861.34 + 445.67 - 1163
= 144.01

<table>
<thead>
<tr>
<th>Source</th>
<th>d.f.</th>
<th>S.S.</th>
<th>M.S.</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Blocks (unadj.)</td>
<td>3</td>
<td>445.67</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Treatments (adj.)</td>
<td>3</td>
<td>861.34</td>
<td>287.11</td>
<td>6.75</td>
</tr>
<tr>
<td>Blocks (adj.)</td>
<td>3</td>
<td>144.01</td>
<td>48.00</td>
<td>1.13</td>
</tr>
<tr>
<td>Treatments (unadj.)</td>
<td>3</td>
<td>1163.00</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Tabled Value F (3,5) = 9.0135  (at 5% level of significance)
Treatment effects are not significant.

\[ CD = t_{0.05} \times \left[ 2k \frac{MSE}{(\lambda v)} \right]^{1/2} = 2.57 \sqrt{\frac{2 \times 3}{2 \times 4}} \times 42.40 \]

\[ = 14.4926 \]

<table>
<thead>
<tr>
<th></th>
<th>( \bar{T}_1 )</th>
<th>( \bar{T}_2 )</th>
<th>( \bar{T}_3 )</th>
<th>( \bar{T}_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Adjusted Treatm</td>
<td>72.50</td>
<td>73.13</td>
<td>61.00</td>
<td>51.37</td>
</tr>
<tr>
<td>Means:</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\( \bar{T}_1 - \bar{T}_2 = -0.63 \)
\( \bar{T}_1 - \bar{T}_3 = 11.5 \)
\( \bar{T}_1 - \bar{T}_4 = 21.13 \) \( T_1 \) is significantly different from \( T_4 \)
\( \bar{T}_2 - \bar{T}_3 = 12.13 \)
\( \bar{T}_2 - \bar{T}_4 = 21.76 \) \( T_2 \) is significantly different from \( T_4 \)
\( \bar{T}_3 - \bar{T}_4 = 9.63 \)

References